PH 5840 Lecture 9 21/8/2015

Note Title 8/20/2015

* Physical motivation for partial trace:-

- The reduced state must reproduce the statistics for local measurements.

(i) Local observable $\longrightarrow$ global observable.

Say M_A is some local observable on system A.
$\tilde{M}_{AB}$ is the corresponding global measurement.

Given $M_A \equiv \{ m, P_m \}_{m=1,2,\ldots \dim(\mathcal{H}_A)}$

- If state in $\mathcal{H}_A$ is $|m\rangle_A$, measurement of M_A should give outcome 'm' with probability one.
- Same statistics for $\tilde{M}_{AB}$

$\Rightarrow$ If $|\Psi\rangle_{AB} = |m\rangle_A |\phi\rangle_B$, measurement $\tilde{M}_{AB}$ should also give outcome 'm', with probability one, independent of $|\phi_B\rangle \in \mathcal{H}_B$.

$$\tilde{M}_{AB} = m |m\rangle_A |\phi\rangle_B \; , \; \forall \; |\phi\rangle_B \in \mathcal{H}_B .$$

$$\Rightarrow \tilde{M}_{AB} \equiv \sum_m m P_m \otimes I_B = \underline{\underline{M_A \otimes I_B}}$$

(ii) If $\rho_A \in \mathcal{H}_A$ is the local state corresponding to $\rho_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$,

$$\mathrm{Tr}_A(\rho_A M_A) = \mathrm{Tr}_{AB}(\rho_{AB} \tilde{M}_{AB}) \quad \text{(Same statistics!)}$$

$$= \mathrm{Tr}_{AB}\left(\rho_{AB}(M_A \otimes I_B)\right)$$

$$= \mathrm{Tr}_{AB}\left(\rho_{AB}\left(M_A \otimes \sum_j |j\rangle\langle j|_B\right)\right)$$

$$= \sum_j \mathrm{Tr}_{AB}\left[\rho_{AB}\left(M_A \otimes |j\rangle\langle j|_B\right)\right]$$

Let $\rho_{AB} = \sum_{i,k} c_{ik}\left(|i_A\rangle|k_B\rangle\langle i_A|\langle k_B|\right)$

$$\therefore \mathrm{Tr}_A(\rho_A M_A) = \sum_{ik} c_{ik}\,\mathrm{Tr}_{AB}\left(|i\rangle_A\langle i|_A M_A \otimes |k\rangle\langle k|\right)$$

$$= \sum_i c_{ik}\,\mathrm{Tr}_A\left(|i\rangle_A\langle i|_A M_A\right)$$

$$= \mathrm{Tr}_A\left[\left(\sum_i c_{ik}|i_A\rangle\langle i|_A\right)M_A\right]$$

$$\therefore \rho_A = \sum_i c_{ik}|i_A\rangle\langle i_A|$$

$$= \langle j_B|\left(\sum_{i,k} c_{ik}|i_A k_B\rangle\langle i_A k_B|\right)|j_B\rangle$$

$$= \underline{\underline{\mathrm{Tr}_B(\rho_{AB})}} \;!$$

(iii) Partial trace is the unique mapping

$$f(\rho_{AB}) = \rho_A$$

satisfying $\mathrm{Tr}\left(M_A f(\rho_{AB})\right) = \mathrm{Tr}\left[(M_A \otimes I_B)\rho_{AB}\right]$

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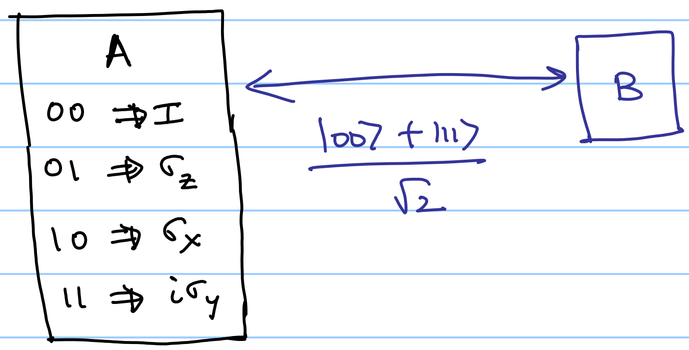
* **Super-dense coding:-**

Communicate 2 c-bits of information using 1 qubit and 1 e-bit (1 entangled state!)



* Quantifying Entanglement : Schmidt decomposition

Which state is more entangled?

$$\begin{cases} |\Psi\rangle = \dfrac{|00\rangle + |01\rangle + |11\rangle}{\sqrt{3}} \\ |\Phi\rangle = \dfrac{|00\rangle + |11\rangle}{\sqrt{2}} \end{cases}$$

Theorem:- Given a pure state $|\psi\rangle_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$, $\exists$ ON states $\{|e_i\rangle_A\} \in \mathcal{H}_A$ and $\{|f_j\rangle_B\} \in \mathcal{H}_B$ such that,

$$|\psi\rangle_{AB} = \sum_{i=1}^{k} \lambda_i |e_i\rangle_A |f_i\rangle_B ,$$

where $\{\lambda_i\}$ are non-negative real nos. Satisfying $\sum_i \lambda_i^2 = 1$

$\{\lambda_i\} \rightarrow$ Schmidt coefficients.

Proof:- Let $\{|j\rangle_A\} \in \mathcal{H}_A$, $\{|k\rangle_B\} \in \mathcal{H}_B$ be ON bases for $\mathcal{H}_A$ and $\mathcal{H}_B$.

Then, $$|\psi\rangle_{AB} = \sum_{j=1}^{d} \sum_{k=1}^{d} c_{jk} |j\rangle_A |k\rangle_B .$$

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* Singular value decomposition of $C \equiv c_{jk}$.

$\exists$ unitary matrices U, V such that

$$C = U \underbrace{D}_{\text{diagonal}} V$$

* Polar decomposition : $C = \underbrace{W}_{\text{unitary}} \underbrace{M}_{\sqrt{C^\dagger C}}$ (positive matrix)

$$\therefore C = WM = WTDT^\dagger \equiv UDV!$$

Proof of polar decomposition:- $C \in \mathcal{L}(\mathcal{H})$

Consider $M = \sqrt{C^\dagger C} \xrightarrow[\text{diagonalize}]{} M = \sum_i m_i |m_i\rangle\langle m_i|$

Let $|\phi_i\rangle = \frac{C|m_i\rangle}{m_i}$ ($\forall$ non-zero eigenvalues)

$\therefore \langle \phi_i | \phi_j \rangle = \frac{\langle m_j | C^\dagger C | m_i \rangle}{m_i m_j} = \delta_{ij}$

- Extend $\{|\phi_i\rangle\}$ using Gram-Schmidt to get an ON basis for $\mathcal{H}$.

- Define unitary $W : \{m_i\} \longrightarrow \{|\phi_j\rangle\}$

i.e. $W = \sum_i |\phi_i\rangle\langle m_i|$

Now, $\forall\ m_i \neq 0$, $WM|m_i\rangle = m_i|\phi_i\rangle = C|m_i\rangle$

$m_i = 0$, $WM|m_i\rangle = 0|m_i\rangle \equiv |\phi_i\rangle$

$\therefore \forall \{|m_i\rangle\}$, $WM|m_i\rangle = C|m_i\rangle \Rightarrow \underline{\underline{C = WM}}$

* Back to Schmidt decomposition!

$$|\psi\rangle_{AB} = \sum_{jk} C_{jk} |j\rangle_A |k\rangle_B$$

$$C_{jk} = \sum_i u_{ji} d_{ii} v_{ik} \quad (\text{SVD})$$

$$\therefore |\psi\rangle_{AB} = \sum_{ijk} u_{ji} d_{ii} v_{ik} |j\rangle_A |k\rangle_B$$

$$= \sum_i d_{ii} \left[\left(\sum_j u_{ji} |j\rangle_A \right) \left(\sum_k v_{ik} |k\rangle_B \right) \right]$$

Define $|e_i\rangle_A \equiv \sum_j u_{ji} |j\rangle_A$, $|f_i\rangle_B \equiv \sum_k v_{ik} |k\rangle_B$

$$\langle e_l | e_i \rangle = \delta_{il} = \langle f_l | f_i \rangle$$

by unitarity of U and V.

$$\therefore \quad |\psi\rangle_{AB} = \sum_i d_{ii} |e_i\rangle_A |f_i\rangle_B ,$$

$\sum_i d_{ii} = 1$ follows from $\langle \psi_{AB} | \psi_{AB} \rangle = 1$.

Usefulness of Schmidt decomposition:-

(i) Reduced density operators:-

$$\rho_A = \sum_i \lambda_i^2 |i\rangle\langle i| \qquad \rho_B = \sum_j \lambda_j^2 |j\rangle\langle j|$$

Have identical eigenvalues!

(ii) # of non-zero Schmidt coefficients

$\equiv$ Schmidt number

A measure of how entangled the state is.